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Free convection at low Prandtl numbers

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In this paper it is shown that the free convection boundary layer approaches a singular character if the Prandtl number tends to zero. The method of matched asymptotic expansions is used to integrate the equations for this extreme case. An expression is derived for the Nusselt–Grashof relation and the results are compared with those of previous investigations which attack the problem in a different way.

1. Introduction

The study of free convection under low Prandtl number conditions has received the attention of a considerable number of authors. The reason for this is quite obvious, as low Prandtl numbers are realized by liquid metals which are known to have a large coefficient of heat conduction. The free convection cooling with liquid metals has important applications, e.g. in nuclear reactors.

The methods of solution employed previously may be divided into two classes. The first class of methods is concerned with the integration of the original partial differential equations by means of the integral method of Kármán–Pohlhausen. Examples may be found in Eckert (1950) and Braun & Heighway (1960). The advantageous side of this method is that it gives results which display the Prandtl number explicitly. Disadvantages are its inaccuracy and its inherent systematic errors. In the other class of methods ordinary differential equations, which are derived from the original equations through a similarity transformation, are integrated by means of an electronic computer (Ostrach 1953; Sparrow & Gregg 1958*a*). This may yield exact results; however, for each Prandtl number a separate integration of the differential equations has to be performed. The results, instead of being represented in one single formula, have to be tabulated. Another disadvantage of the method of the second class is the fact that for low Prandtl numbers the equations contain coefficients of different orders of magnitude. Numerical integration (Sparrow & Gregg 1958*a*) shows that this greatly affects the velocity profiles. In a restricted region the velocity gradients are much larger than in the remaining part of the boundary layer. It is clear that this behaviour considerably encumbers numerical integration.

Thus it would be desirable to have a method which is free of the objectionable features of the two classes, but which maintains the advantages of each, i.e

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accuracy and explicit Prandtl number. In this paper such a method is developed using a singular perturbation technique of the type described in the book of Van Dyke (1964). Apart from having the required characteristics just mentioned the analysis will reveal much of the anatomy of a boundary layer of free convection at low Prandtl number. This is because the method of matched asymptotic expansions exposes the predominant factors in different parts of the boundary layer. It goes without saying that the knowledge of these factors at low Prandtl numbers is of great theoretical interest. The analysis and its results are compared with previous investigations.

2. Main terms of inner and outer expansions

It is well known that free convection boundary-layer flow past a vertical isothermal flat plate is governed by a set of two coupled differential equations (Ostrach 1953),

$$\frac{d^3f}{d\eta^3}+3f\frac{d^2f}{d\eta^2}-2\left(\frac{df}{d\eta}\right)^2+\theta = 0, \tag{2.1}$$

$$\frac{d^2\theta}{d\eta^2}+3\sigma f\frac{d\theta}{d\eta} = 0. \tag{2.2}$$

The functions f and θ, which are the non-dimensional stream function and temperature respectively, have to satisfy the boundary conditions

$$f = df/d\eta = 0, \quad \theta = 1 \quad \text{at} \quad \eta = 0, \tag{2.3}$$

$$df/d\eta \to 0, \quad \theta \to 0 \quad \text{as} \quad \eta \to \infty. \tag{2.4}$$

η is the free convection similarity variable and σ is the Prandtl number

$$\sigma = \frac{\nu\rho c_p}{k}. \tag{2.5}$$

Here ν is the kinematic viscosity, ρ the density, c_p the specific heat and k the thermal conductivity.

An investigation into the limit of a free convection boundary layer for $\sigma \to 0$ can be developed along two different lines. Braun & Heighway (1960) have discussed some aspects of this matter in order to develop a suitable Kármán–Pohlhausen method for low Prandtl numbers. For the purpose of stating the correct boundary-layer conditions, the problem was revisited by Kuiken (1967, 1968). The first possibility is to investigate the equations (2.1)–(2.4) on a purely mathematical basis. Since the different equations involved are strongly nonlinear, a second way of attacking the problem is advocated in this paper.

From the numerical solution for $\sigma = 0{\cdot}003$ of Sparrow & Gregg (1958*a*), shown in figure 4 (solid line), it can be seen that the velocity distribution for small Prandtl number can be considered to consist of two distinct parts. Near the wall there is a thin region of large velocity gradients due to viscous effects. In the remaining, much wider, region the gradients are small compared with those near the wall; in this region the fluid can be considered to be inviscid.

This viscous boundary layer can be obtained easily from (2.1) and (2.2). Suppose that ν is constant and that $k \to \infty$. It is then clear that $\sigma \to 0$. Since the conductivity k becomes infinitely large, it is obvious that the temperature of the wall is maintained in the fluid. Thus near the wall we have $\theta = 1$. As the general solution of (2.2) for $\sigma = 0$ is

$$\theta = a + b\eta, \tag{2.6}$$

this solution can be obtained from the equations by putting $a = 1$ and $b = 0$.

Another aspect of this limiting process emerges if we consider that due to the horizontal temperature distribution, the fluid experiences a uniform force in an upward direction. Consequently at some distance from the plate, where the viscous stresses may be neglected, the flow has a uniform acceleration in an upward direction. This means that there exists a potential flow U which is proportional to $x^{\frac{1}{2}}$. This follows directly from an elementary law of mechanics. So, for $\sigma = 0$, the viscous part of the free convection boundary layer is related to the Falkner–Skan wedge-flow with a potential flow $U = px^{\frac{1}{2}}$. This result has been reported briefly already by Lykoudis (1962).

The considerations advanced up to now make us decide that there is an inner expansion,

$$f_{\text{inner}} = f_0(\eta) + \epsilon_1(\sigma) f_1(\eta) + \epsilon_2(\sigma) f_2(\eta) + \dots, \tag{2.7}$$

$$\theta_{\text{inner}} = 1 + \bar{\epsilon}_1(\sigma)\,\theta_1(\eta) + \bar{\epsilon}_2(\sigma)\,\theta_2(\eta) + \dots, \tag{2.8}$$

which is valid in the vicinity of the plate. The boundary conditions at the surface (2.3) apply to the functions of this expansion. The main term of the inner expansion f_0 naturally has to satisfy the differential equation

$$\frac{d^3 f_0}{d\eta^3} + 3 f_0 \frac{d^2 f_0}{d\eta^2} - 2\left(\frac{df_0}{d\eta}\right)^2 + 1 = 0. \tag{2.9}$$

For the expansion parameters $\epsilon_i(\sigma)$ and $\bar{\epsilon}_i(\sigma)$, the following conditions hold:

$$\lim_{\sigma \downarrow 0} \frac{\epsilon_{i+1}}{\epsilon_i} = 0, \quad \lim_{\sigma \downarrow 0} \frac{\bar{\epsilon}_{i+1}}{\bar{\epsilon}_i} = 0, \quad \epsilon_0 = \bar{\epsilon}_0 = 1. \tag{2.10}$$

For the integration of (2.9) we lack one boundary condition, since the inner boundary conditions (2.3) only supply $f_0 = df_0/d\eta = 0$ at $\eta = 0$. The third boundary condition will be found later through matching with an outer expansion to be derived next.

It has been remarked above that the major part of the low-Prandtl-number boundary layer of free convection is inviscid. Free convection of an inviscid fluid has been studied by Lefevre (1956). We may anticipate that the first term of our outer expansion will be a solution of Lefevre's equations. Introducing the new variables

$$\xi = 3(18)^{-\frac{1}{4}} \eta \sigma^{\frac{1}{2}}, \tag{2.11}$$

$$f_{\text{outer}} = 18^{-\frac{1}{4}} \sigma^{-\frac{1}{2}} F(\xi), \tag{2.12}$$

$$\theta_{\text{outer}} = \vartheta(\xi), \tag{2.13}$$

the equations (2.1) and (2.2) are transformed into

$$\tfrac{3}{2}\sigma \frac{d^3 F}{d\xi^3} + \tfrac{3}{2} F \frac{d^2 F}{d\xi^2} - \left(\frac{dF}{d\xi}\right)^2 + \vartheta = 0, \tag{2.14}$$

$$\frac{d^2\vartheta}{d\xi^2} + F\frac{d\vartheta}{d\xi} = 0. \tag{2.15}$$

Indeed, on taking $\sigma = 0$, the viscous term $d^3F/d\xi^3$ is left out of (2.14) as it should be. The forces due to inertia and those due to buoyancy balance each other. The flow is not hampered by any viscous stresses. Consequently (2.14) and (2.15) are suitable for the description of the outer part of the free convection boundary layer. The boundary conditions to be satisfied by the outer functions naturally are those depicting the ambient conditions (2.4). So we have

$$dF/d\xi \to 0, \quad \vartheta(\xi) \to 0 \quad \text{as} \quad \xi \to \infty. \tag{2.16}$$

The remaining boundary conditions have to be found through matching with the inner expansion. The outer expansions to be analyzed are given by

$$F(\xi) = F_0(\xi) + \delta_1(\sigma) F_1(\xi) + \delta_2(\sigma) F_2(\xi) + \ldots, \tag{2.17}$$

$$\vartheta(\xi) = \vartheta_0(\xi) + \bar{\delta}_1(\sigma) \vartheta_1(\xi) + \bar{\delta}_2(\sigma) \vartheta_2(\xi) + \ldots, \tag{2.18}$$

with

$$\lim_{\sigma \downarrow 0} \frac{\delta_{i+1}}{\delta_i} = 0, \quad \lim_{\sigma \downarrow 0} \frac{\bar{\delta}_{i+1}}{\bar{\delta}_i} = 0, \quad \delta_0 = \bar{\delta}_0 = 1. \tag{2.19}$$

Upon insertion of the outer expansions (2.17) and (2.18) in the equations (2.14) and (2.15) F_0 and ϑ_0 indeed turn out to satisfy the equations of Lefevre,

$$\tfrac{3}{2} F_0 \frac{d^2F_0}{d\xi^2} - \left(\frac{dF_0}{d\xi^2}\right)^2 + \vartheta_0 = 0, \tag{2.20}$$

$$\frac{d^2\vartheta_0}{d\xi^2} + F_0 \frac{d\vartheta_0}{d\xi} = 0. \tag{2.21}$$

Now, according to the matching principle (Van Dyke 1964), we have to impose the conditions

$$\lim_{\eta \to \infty} f_{\text{inner}}(\eta) = \lim_{\xi \downarrow 0} f_{\text{outer}} = \lim_{\xi \downarrow 0} 18^{-\frac{1}{4}} \sigma^{-\frac{1}{2}} F(\xi), \tag{2.22}$$

$$\lim_{\eta \to \infty} \theta_{\text{inner}}(\eta) = \lim_{\xi \downarrow 0} \theta_{\text{outer}} = \lim_{\xi \downarrow 0} \vartheta(\xi), \tag{2.23}$$

for $\sigma \downarrow 0$. Here lim stands for the behaviour of a function in the direction given in the formula. In the present analysis, these conditions can be applied best by writing (2.17) and (2.18) in terms of the inner variable η and expanding the expression for small values of σ. Comparison with the asymptotic representations ($\eta \to \infty$) of f_{inner} and θ_{inner} then supplies the remaining boundary conditions. Now, assuming that F_0 and ϑ_0 can be expanded as series of increasing powers of ξ as

$$F_0 = F_0(0) + \text{positive powers of } \xi, \tag{2.24}$$

$$\vartheta_0 = \vartheta_0(0) + \text{positive powers of } \xi, \tag{2.25}$$

the expansions for small values of σ yield (use is made of (2.11), (2.12) and (2.13))

$$f_{\text{outer}} = 18^{-\frac{1}{4}} \sigma^{-\frac{1}{2}} F_0(0) + O(\sigma^r) \quad (r > -\tfrac{1}{2}), \tag{2.26}$$

$$\theta_{\text{outer}} = \vartheta_0(0) + O(\sigma^s) \quad (s > 0). \tag{2.27}$$

From the fact that the inner expansion (2.7) does not contain any terms with a negative power of σ we may infer, using (2.22), that

$$F_0(0) = 0. \tag{2.28}$$

Comparison of the inner expansion for θ (2.8) with (2.27) gives, according to (2.23),

$$\vartheta_0(0) = 1. \tag{2.29}$$

In fact, these boundary conditions were also used by Lefevre.

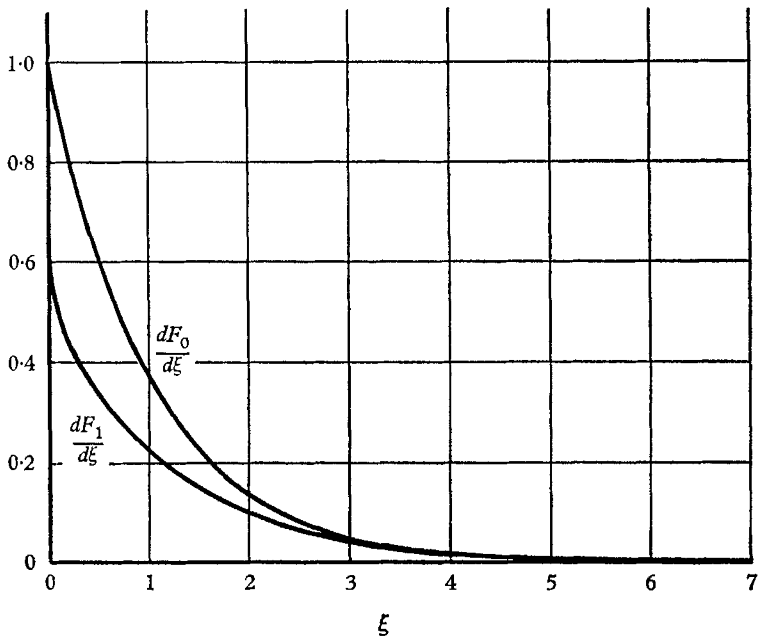

FIGURE 1. Perturbation functions of outer expansion (velocity).

Hence the zeroth perturbation of the outer expansion must satisfy the system of equations (2.20) and (2.21) with the boundary conditions (2.16), (2.28) and (2.29). A series solution for F_0 and ϑ_0 valid near $\xi = 0$ is (see appendix)

$$F_0 = \xi + a_0\xi^2 + b_0\xi^{\frac{7}{3}} + \frac{a_0^2}{3}\xi^3 + \frac{a_0 b_0}{3}\xi^{\frac{10}{3}} + \tfrac{7}{66}b_0^2\xi^{\frac{11}{3}} + \frac{a_0}{60}\xi^4 - \frac{a_0^2 b_0}{39}\xi^{\frac{13}{3}} + O(\xi^{\frac{14}{3}}), \tag{2.30}$$

$$\vartheta_0 = 1 + a_0\xi - \frac{a_0}{6}\xi^3 - \frac{a_0^2}{12}\xi^4 - \frac{9}{130}a_0 b_0\xi^{\frac{13}{3}} + O(\xi^{\frac{14}{3}}), \tag{2.31}$$

with

$$a_0 = -0{\cdot}582983, \tag{2.32}$$

$$b_0 = 0{\cdot}13744. \tag{2.33}$$

Another quantity of importance is

$$F_0(\infty) = 1{\cdot}007366. \tag{2.34}$$

Application of the matching rule (2.22) to (2.7), (2.17) and (2.30) gives

$$\lim_{\eta\to\infty} f_0(\eta) = \tfrac{1}{2}\eta\sqrt{2} + \text{lower-order terms of } \eta,$$

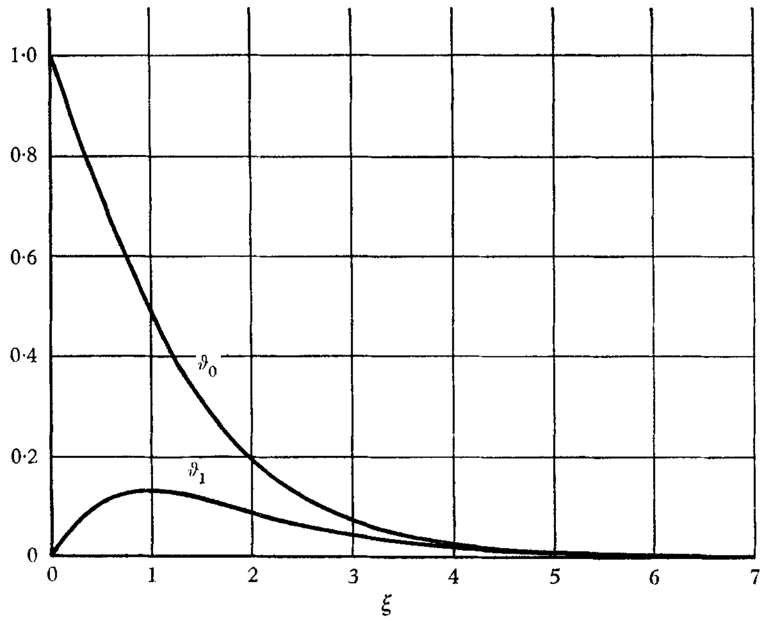

FIGURE 2. Perturbation functions of outer expansion (temperature).

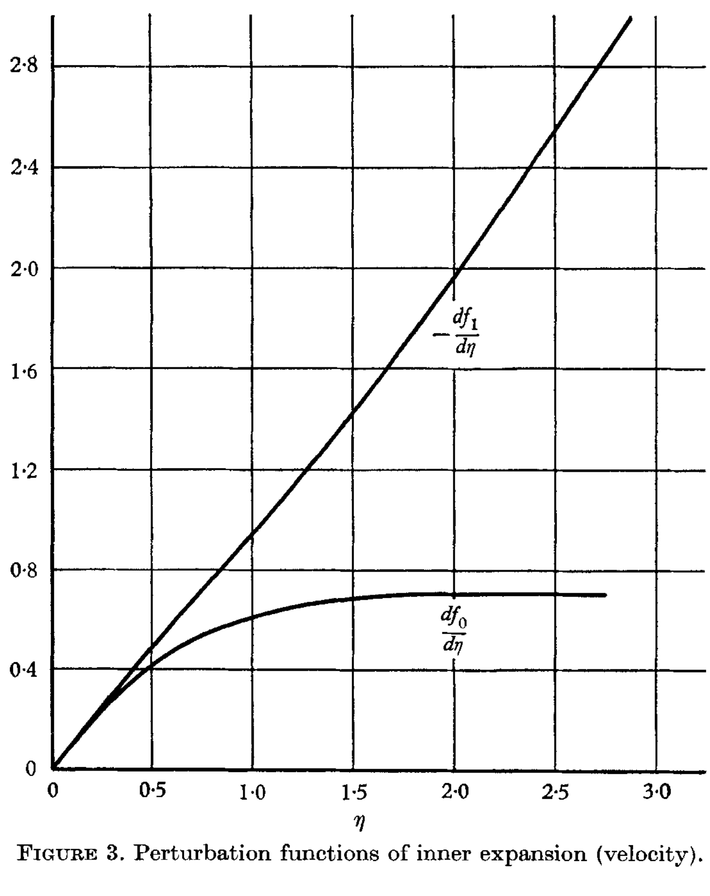

FIGURE 3. Perturbation functions of inner expansion (velocity).

which may be written

$$\frac{df_0}{d\eta} \to \tfrac{1}{2}\sqrt{2} \quad \text{as} \quad \eta \to \infty. \tag{2.35}$$

Together with the wall conditions (2.3), equation (2.35) provides a complete set of boundary conditions for the integration of (2.9), which yields

$$\frac{d^2f_0}{d\eta^2} = 1{\cdot}069950 \quad \text{at} \quad \eta = 0, \tag{2.36}$$

$$f_0(\eta) \sim 0{\cdot}707107\eta - 0{\cdot}359348 + \text{exp}- \quad \text{as } \eta \to \infty. \tag{2.37}$$

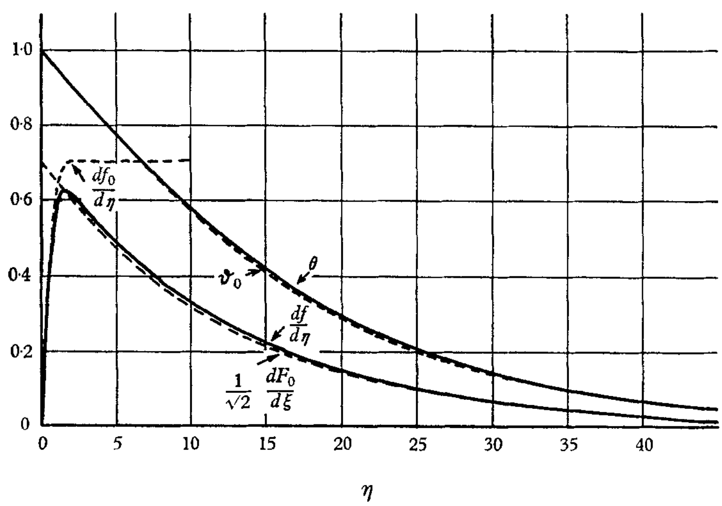

FIGURE 4. Comparisons of Sparrow & Gregg's (1958*a*) solution (σ = 0·003) (continuous curves) with one-term inner and outer expansions (dashed lines).

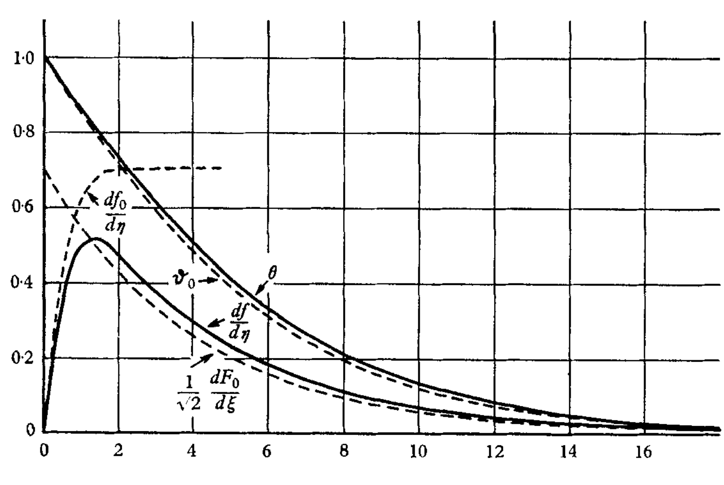

FIGURE 5. Comparison of Sparrow & Gregg's (1958*a*) solution (σ = 0·03) (continuous curves) with one-term inner and outer expansions (dashed lines).

Here exp– stands for terms of exponentially small order. Graphs of the main terms of the inner and outer expansions may be found in figures 1–3.

Before proceeding to the terms of higher order it is interesting to compare the results derived up to now with those of the literature. In figures 4 and 5, the continuous curves are results of a numerical integration of Sparrow & Gregg (1958*a*) for $\sigma = 0{\cdot}003$ and $\sigma = 0{\cdot}03$, while the dashed curves are based on the present analysis.

It is clear that the inner expansion covers that part of the boundary layer where large velocity gradients exist; the outer expansions refer to the small velocity gradients. The predominant part of the free convection boundary layer is inviscid and tends to create a uniformly accelerated flow at the surface. However, on coming within the range of the viscous stresses a viscous boundary layer of forced flow type is created, which in the limiting case of $\sigma = 0$ is exactly the same as a particular Falkner–Skan boundary layer.

3. First perturbations

In view of the fact that the first terms of the expansions (2.30) and (2.31) are integer powers of ξ we infer that the expansion variables $\epsilon_1(\sigma)$, $\bar{\epsilon}_1(\sigma)$, $\delta_1(\sigma)$, $\bar{\delta}_1(\sigma)$ are all equal to $\sigma^{\frac{1}{2}}$.

Insertion of the inner and outer expansions (2.7), (2.8), (2.17) and (2.18) in their respective systems of differential equations (2.1), (2.2) and (2.14), (2.15) two systems pertaining to the first perturbations follow. For the inner problem we have

$$\frac{d^3f_1}{d\eta^3}+3f_0\frac{d^2f_1}{d\eta^2}-4\frac{df_0}{d\eta}\frac{df_1}{d\eta}+3\frac{d^2f_0}{d\eta^2}f_1+\theta_1 = 0, \tag{3.1}$$

$$\frac{d^2\theta_1}{d\eta^2} = 0, \tag{3.2}$$

while for the outer problem the differential equations are

$$\tfrac{3}{2}F_0\frac{d^2F_1}{d\xi^2}-2\frac{dF_0}{d\xi}\frac{dF_1}{d\xi}+\frac{3}{2}\frac{d^2F_0}{d\xi^2}F_1+\vartheta_1 = 0, \tag{3.3}$$

$$\frac{d^2\vartheta_1}{d\xi^2}+F_0\frac{d\vartheta_1}{d\xi}+\frac{d\vartheta_0}{d\xi}F_1 = 0. \tag{3.4}$$

If we first pay attention to the temperature, it is seen from (3.2) that

$$\theta_1 = \alpha_1\eta. \tag{3.5}$$

Here we have used the inner boundary condition $\theta_1(0) = 0$. The two-term inner expansion for the temperature now is

$$\theta_{\text{inner}} = 1+\alpha_1\eta\sigma^{\frac{1}{2}}. \tag{3.6}$$

Obviously the asymptotic behaviour of θ_{inner} for $\eta\to\infty$ is also described by (3.6). If for a moment we assume $\vartheta_1(0) \neq 0$ a two-term outer expansion written in inner variables for $\sigma \downarrow 0$ gives

$$\theta_{\text{outer}} = 1+(3a_0 18^{-\frac{1}{4}}\eta+\vartheta_1(0))\,\sigma^{\frac{1}{2}}. \tag{3.7}$$

Comparison of (3.6) and (3.7) gives according to the matching principle (2.23)

$$\alpha_1 = 3a_0\,18^{-\frac{1}{4}} = -0{\cdot}84910 \tag{3.8}$$

and

$$\vartheta_1(0) = 0. \tag{3.9}$$

The general solutions of (3.3) and (3.4) which satisfy the condition (3.9) can be represented near $\xi = 0$, as

$$F_1 = a_1 + \tfrac{3}{2}a_0 a_1 \xi + \tfrac{7}{3}a_1 b_0 \xi^{\frac{4}{3}} + (\tfrac{3}{2}a_0 a_1 + b_1)\,\xi^2 + c_1\xi^{\frac{7}{3}} + \tfrac{7}{45}a_1 b_0^2 \xi^{\frac{8}{3}} + O(\xi^3), \tag{3.10}$$

$$\vartheta_1 = b_1\xi - \tfrac{1}{2}a_0 a_1 \xi^2 - (\tfrac{1}{4}a_0^2 a_1 + \tfrac{1}{6}b_1)\,\xi^3 + O(\xi^{\frac{10}{3}}). \tag{3.11}$$

Writing a two-term inner expansion and a two-term outer expansion for f as required by the matching principle

$$\lim_{\eta\to\infty} f_{\text{inner}} \sim \tfrac{1}{2}\eta\sqrt{(2)} - 0{\cdot}359348 + \sigma^{\frac{1}{2}} f_1(\eta \to \infty), \tag{3.12}$$

$$\lim_{\xi\downarrow 0} f_{\text{outer}} \sim \tfrac{1}{2}\eta\sqrt{(2)} + 18^{-\frac{1}{4}}a_1 + \sigma^{\frac{1}{2}}\{9(18)^{-\frac{3}{4}}a_0\eta^2 + \text{lower-order terms of } \eta\}, \tag{3.13}$$

it follows that

$$a_1 = -(0{\cdot}359348)\,(18)^{\frac{1}{4}} = -0{\cdot}740174 \tag{3.14}$$

and

$$\frac{d^2 f_1}{d\eta^2} \to 18^{\frac{1}{4}}a_0 = -1{\cdot}20081 \quad \text{as} \quad \eta \to \infty. \tag{3.15}$$

Among the lower-order terms of η in (3.13), a term occurs which originates from the expansion of F_1 (3.10). This is the term

$$\tfrac{3}{4}a_0 a_1 \eta\sqrt{(2)} = 0{\cdot}45768\eta. \tag{3.16}$$

The asymptotic behaviour of $f_1(\eta)$ was found to be, after integrating (3.1) with the inner boundary conditions (2.3) and the one found through matching (3.15),

$$f_1(\eta) \sim -0{\cdot}600405\eta^2 + 0{\cdot}45768\eta - 0{\cdot}4385 + \text{exp-} \quad \text{as} \quad \eta \to \infty. \tag{3.17}$$

It is evident that once the condition (3.15) has been imposed the term (3.16) appears automatically in the asymptotic behaviour (3.17).

The fact that both coefficients of η in (3.16) and (3.17) are results of completely different integrations gives a valuable means of determining the accuracy of the numerical work. Apparently at least five decimal places are significant.

For the skin friction an important figure is

$$\frac{d^2 f_1}{d\eta^2} = -1{\cdot}001023 \quad \text{at} \quad \eta = 0. \tag{3.18}$$

The system of equations (3.3) and (3.4) is to be integrated with the outer boundary conditions (2.16) and with $F_1(0) = a_1$—((3.10) and (3.14))—and $\vartheta_1(0) = 0$—(3.9). This integration supplies the numerical values of the remaining constants b_1 and c_1 of the expansions (3.10) and (3.11).

For subsequent use, the value of b_1 will be given here:

$$b_1 = 0{\cdot}31445. \tag{3.19}$$

Another quantity of importance is

$$F_1(\infty) = -0{\cdot}10867. \tag{3.20}$$

Graphs of the first perturbations can be found in figures 1–3.

4. Second perturbations

Inspection of the expansions (2.30) and (3.10) shows that the second perturbation is influenced by such terms as $\xi^{\frac{7}{3}}$, $\xi^{\frac{4}{3}}$, etc. A three-term outer expansion for f and θ gives for the term following $\sigma^{\frac{1}{2}}$

$$f_{\text{outer}} = \sigma^{\frac{2}{3}}\{2^{-\frac{5}{6}}3^{\frac{2}{3}}b_0\eta^{\frac{7}{3}} + 2^{-\frac{7}{12}}3^{-\frac{5}{6}}7a_1b_0\eta^{\frac{4}{3}} + \text{lower-order terms of } \eta\}. \tag{4.1}$$

As a consequence the expansion variable $\epsilon_2(\sigma)$ of the inner expansion should be $\sigma^{\frac{2}{3}}$. Consideration of an expression for θ_{outer} analogous to (4.1) suggests that $\bar{\epsilon}_2(\sigma)$ be merely σ as the fractional powers enter the outer expansion for the temperature in a much later stage. So, for the second perturbation of the inner expansion, we merely have a differential equation for $f_2(\eta)$

$$\frac{d^3f_2}{d\eta^3} + 3f_0\frac{d^2f_2}{d\eta^2} - 4\frac{df_0}{d\eta}\frac{df_2}{d\eta} + 3\frac{d^2f_0}{d\eta^2}f_2 = 0. \tag{4.2}$$

On account of the matching principle this equation must have an asymptotic behaviour coinciding with the expression between brackets of (4.1). Using the asymptotic expansion (2.37) of f_0 the solution of (4.2) for large values of η has to satisfy the equation

$$\frac{d^3f_2}{d\eta^3} + 3(\tfrac{1}{2}\eta\surd(2) - p)\frac{d^2f_2}{d\eta^2} - 2\surd 2\frac{df_2}{d\eta} = 0. \tag{4.3}$$

Here terms of exponentially small order have been omitted and p is the constant 0·359348. This equation is known to have the general solution (Morse & Feshbach 1953)

$$\frac{df_2}{d\eta} = AH[-\tfrac{2}{3}, \tfrac{1}{2}; \mu^2] + B\mu H[-\tfrac{1}{6}, \tfrac{3}{2}; \mu^2], \tag{4.4}$$

where $\eta = 2^{\frac{3}{4}}3^{-\frac{1}{2}}i\mu + 2^{\frac{1}{2}}p$ and H is the confluent hypergeometric function. Using the asymptotic expansion of the confluent hypergeometric function (Slater 1960)

$$H[a,b;\ -x] \sim x^{-a}\frac{\Gamma(b)}{\Gamma(b-a)}\sum_{n=0}^{L}\frac{(a)_n(1+a-b)_n}{n!}x^{-n} + O(x^{-a-L-1}) \quad \text{as} \quad x \to \infty, \tag{4.5}$$

it can be shown that the highest-order terms of $df_2/d\eta$ for $\eta \to \infty$ are

$$(\eta - p\surd 2)^{\frac{4}{3}}, \quad (\eta - p\surd 2)^{\frac{1}{3}}, \text{ etc.} \tag{4.6}$$

It is thus proved that matching with the outer expansion (4.1) is possible. We confine ourselves to these qualitative remarks about the second perturbation of the stream function, since our main concern is the problem of heat transfer for which numerical results are important.

With respect to heat transfer it is possible to derive an important result from the values obtained above. Proceeding along exactly the same lines it can be shown that the expansion variables to be used now are

$$\epsilon_3(\sigma) = \bar{\epsilon}_2(\sigma) = \delta_2(\sigma) = \bar{\delta}_2(\sigma) = \sigma.$$

The inner expansion for the temperature then yields a solution

$$\theta_2(\eta) = \alpha_2\eta. \tag{4.7}$$

An evaluation of θ_{outer} at the wall gives for the coefficient of σ

$$3(18)^{-\frac{1}{4}}b_1\eta + \text{lower-order terms of } \eta. \tag{4.8}$$

Thus the constant α_2 in (4.7) is proved to be

$$\alpha_2 = 3(18)^{-\frac{1}{4}}b_1 = 0{\cdot}45799. \tag{4.9}$$

With respect to this, it is interesting to note that the term with ξ^2 is quite naturally left out of the expansion (2.31). If not, such a term would have produced a term with η^2 in (4.8), thus inhibiting the matching.

5. Results

The most important result to be given in problems of the present type is an expression for heat transfer. This can be presented most conveniently by introduction of the local Nusselt number

$$Nu_x = -\frac{x}{T_w - T_\infty}\frac{\partial T}{\partial y}\bigg|_{y=0}, \tag{5.1}$$

and the local Grashof number

$$G_x = \frac{g\beta(T_w - T_\infty)\,x^3}{\nu^2}. \tag{5.2}$$

Here T_w is the wall temperature, T_∞ is the ambient temperature, g is the acceleration due to gravity, β is the coefficient of thermal expansion and x and y are the longitudinal and the normal co-ordinates respectively. These numbers are traditionally combined in one expression which in our case leads to

$$\frac{Nu_x}{G_x^{\frac{1}{4}}} = -\frac{1}{\sqrt{2}}\frac{d\theta}{d\eta}\bigg|_{\eta=0} = 2^{-\frac{1}{2}}\{0{\cdot}84910\sigma^{\frac{1}{2}} - 0{\cdot}45799\sigma + O(\sigma^{\frac{3}{2}})\}, \tag{5.3}$$

or

$$\frac{Nu_x}{(G_x\sigma^2)^{\frac{1}{4}}} = 0{\cdot}60040 - 0{\cdot}32385\sigma^{\frac{1}{2}} + O(\sigma). \tag{5.4}$$

A comparison with the results of Sparrow & Gregg (1958*a*) may be found in table 1.

The expression

$$\frac{d^2f}{d\eta^2}\bigg|_{\eta=0} = 1{\cdot}0699496 - 1{\cdot}001023\sigma^{\frac{1}{2}} + O(\sigma^{\frac{3}{2}}) \tag{5.5}$$

is also of interest, as this quantity is related to the skin friction. Again the values of table 1 allow a comparison with figures of Sparrow & Gregg. It is seen that the figures for heat transfer are in closer agreement than those referring to the skin friction. This is to be expected on account of the different orders of the truncation errors in (5.4) and (5.5).

The coefficients of (5.4) and (5.5) are all related to the inner expansion. It is worth while, therefore, to present an expression for $f(\infty)$ which is related to the outer expansion

$$f(\infty) = 18^{-\frac{1}{4}}\sigma^{-\frac{1}{2}}\{1{\cdot}007366 - 0{\cdot}10867\sigma^{\frac{1}{2}} + O(\sigma)\}. \tag{5.6}$$

σ	$d^2f/d\eta^2\|_{\eta=0}$ Sparrow & Gregg	$d^2f/d\eta^2\|_{\eta=0}$ Present	$Nu_x/(G_x\sigma^2)^{\frac{1}{4}}$ Sparrow & Gregg	$Nu_x/(G_x\sigma^2)^{\frac{1}{4}}$ Present	$f(\infty)$ Sparrow & Gregg	$f(\infty)$ Present
0·003	1·0223	1·0151	0·5827	0·5827	8·7060	8·8763
0·008	0·9955	0·9801	0·5729	0·5714	5·4018	5·4152
0·020	0·9590	0·9284	0·5582	0·5546	3·4093	3·4055
0·030	0·9384	0·8966	0·5497	0·5443	2·7878	2·7710

TABLE 1

6. Concluding remarks

The problem considered in this paper resembles in many respects the already classical investigation into the higher perturbations of two-dimensional forced flow of a viscous fluid along a surface. In the latter problem the non-dimensional boundary-layer equations display the term R^{-1} in the viscous term, R being the Reynolds number. Van Dyke (1962) showed that the interaction of the boundary layer with the inviscid fluid requires the expansion variable $R^{-\frac{1}{2}}$.

The present problem shows the same characteristics. In equation (2.14) the viscous term is multiplied by σ. However, the expansion variable has been shown to be $\sigma^{\frac{1}{2}}$. The first perturbations of the outer expansions consequently do not experience viscous effects through the equations. It is the matching which takes these into account.

In this paper only the vertical isothermal flat plate has been considered. The analysis could of course be repeated for non-isothermal flat plates. It may be in order to note one interesting feature of these more general problems. It has been shown that the main term of the inner layer represents a boundary layer of forced flow type. One might ask what the outer flow is of such a boundary layer for a general wall temperature. Or, in other words, what is the maximum velocity that can occur along a plate through free convection?

The answer may be found by applying Newton's law of motion

$$\frac{d^2x}{dt^2} = g\beta(T_w - T_\infty). \tag{6.1}$$

A first integral of this equation is

$$U = \frac{dx}{dt} = \left\{g\beta\int_0^x (T_w - T_\infty)\,dx\right\}^{\frac{1}{2}}. \tag{6.2}$$

Here the condition that U be zero at the leading edge has been imposed. For a temperature difference of the form

$$T_w = T_\infty + Nx^m, \tag{6.3}$$

which has received considerable attention (Finston 1956; Sparrow & Gregg 1958*a*; Yang 1960), U is seen to be proportional to

$$x^{\frac{1}{2}(m+1)}. \tag{6.4}$$

Thus every power-law wall temperature is related to a certain Falkner–Skan flow. Indeed it can be shown (Kuiken 1967) that for $m > -1$ the inner free convection equation can be converted into the Falkner–Skan equation, thus proving the existence of a free convection boundary layer for $m > -1$. Previously, only numerical calculations (Sparrow & Gregg 1958*b*) made it plausible that there should be a value of m less than $-0{\cdot}8$ for which free convection ceased to be existent.

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Appendix

For matching purposes it is necessary that a series solution of (2.20) and (2.21), valid for small values of ξ, be available. The behaviour of this series solution can be studied most conveniently if we rewrite (2.20) and (2.21) as a system of five first-order equations.

We therefore introduce the new variables

$$x_1(t) = \xi, \quad x_2(t) = F_0, \quad x_3(t) = dF_0/d\xi - 1,$$
$$x_4(t) = \vartheta_0 - 1, \quad x_5(t) = d\vartheta_0/d\xi - c, \tag{A 1}$$

where $c = d\vartheta_0/d\xi|_{\xi=0}$.

The new system is

$$\frac{dx_1}{dt} = \tfrac{3}{2}x_2, \tag{A 2}$$

$$\frac{dx_2}{dt} = \tfrac{3}{2}x_2 + \qquad + \tfrac{3}{2}x_2x_3, \tag{A 3}$$

$$\frac{dx_3}{dt} = \qquad 2x_3 - x_4 + x_3^2, \tag{A 4}$$

$$\frac{dx_4}{dt} = \tfrac{3}{2}cx_2 + \qquad + \tfrac{3}{2}x_2x_5, \tag{A 5}$$

$$\frac{dx_5}{dt} = \qquad -\tfrac{3}{2}x_2^2x_5, \tag{A 6}$$

with the initial conditions $x_i = 0$ for $\xi = 0$. If we consider only the linear approximation, we find from (A 3) and (A 2) $x_1 = x_2 = A\,e^{\frac{3}{2}t}$. Apparently $\xi = x_1 = 0$ for $t = -\infty$. After solving (A 5), the solution of the linear part of (A 4) is seen to be

$$x_3 = r\,e^{\frac{3}{2}t} + s\,e^{2t},$$

or

$$\frac{dF_0}{d\xi} = 1 + x_3 = 1 + r\xi + s\xi^{\frac{4}{3}}. \tag{A 7}$$

Here r is a constant involving c and s is a general constant of integration. This investigation indeed suggests that a series solution for F_0 valid near $\xi = 0$ have the form (2.30). The terms following $\xi^{\frac{7}{3}}$ have been found by successive substitution.

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